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EQUIVALENCE OF LINEAR COMPLEMENTARITY
PROBLEMS AND LINEAR PROGRAMS IN
VECTOR LATTICE HILBERT SPACES.

C. W. Cryer and M. A. H. Dempster

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VECTOR LATTICE HILBERT SPACES

C. W. Cryer[1,2,3] and M. A. H. Dempster[4]

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## ABSTRACT

Let $X$ be a vector lattice Hilbert space with dual $X^*$. Let $M$ be a continuous linear mapping of $X$ onto $X^*$. Let $p, q \in X^*$ with $p > 0$. We consider the relationship between the linear complementarity problem: Find $x \in X$ such that $x \geq 0$, $Mx + q \geq 0$, $\langle x, Mx + q \rangle = 0$, and the linear programming problem: Find $x \in X$ which minimizes $\langle x, p \rangle$ subject to $x \geq 0$, $Mx + q \geq 0$.

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SIGNIFICANCE AND EXPLANATION

Many free boundary problems in the areas of fluid mechanics, porous flow, elasticity, and plasticity can be formulated as linear complementarity problems in which a differential equation (ordinary or partial) must be solved subject to the inequality constraint that the solution be non-negative; roughly speaking, at any point the solution must either be zero or satisfy the differential equation. We study linear complementarity problems which can be reformulated as linear programs in which a linear functional must be minimized subject to inequality constraints. The reformulation of linear complementarity problems as linear programs offers two advantages:

(i)  It suggests alternative numerical methods of solving the problems.

(ii) For the problem of a cavitating journal bearing, which is used as an example, the linear program requires the minimization of a linear functional which is proportional to the load borne by the bearing, so that the linear program has a physical interpretation. It is possible that the linear programs for other problems will also have physical interpretations, though this will have to be determined in each case.

EQUIVALENCE OF LINEAR COMPLEMENTARITY PROBLEMS

AND LINEAR PROGRAMS IN VECTOR LATTICE HILBERT SPACES

C. W. Cryer[1,2,3] and M. A. H. Dempster[4]

## 1. Introduction

The <u>linear complementarity problem</u> in real n-dimensional Euclidean space $R^n$ is: Find $x \in R^n$ such that $x \geq 0$, $Mx + q \geq 0$, and $x^T(Mx - q) = 0$, where $M$ is a given real $n \times n$ - matrix and $q$ is a given vector in $R^n$. The <u>linear programming problem</u> in $R^n$ is: Find $x \in R^n$ which minimizes $p^Tx$ subject to $x \geq 0$ and $Mx + q \geq 0$, where $M$ is a given real $n \times n$ matrix and $p$ and $q$ are given vectors in $R^n$.

Mangasarian [1976] showed that, under certain conditions, each solution of the linear programming problem in $R^n$ is a solution of the linear complementarity problem in $R^n$. Mangasarian [1977, 1977a] has subsequently extended this work. Related work is due to Cottle and Veinott [1972], Moré [1971], Tamir [1973], Cottle, Golub, and Sacher [1974], Cottle and Pang [1976, 1976a], Pang [1977].

Quite independently, and often not very explicitly, the relationship between certain infinite-dimensional linear programming problems and linear complementarity problems has been noted (Moreau [1971], Durand [1968], Lewy and Stampacchia [1969], Stampacchia [1965], Lions and Stampacchia [1967]).

Here, we consider extensions of some of the results of Mangasarian to infinite-dimensional spaces. Apart from their intrinsic value, our results provide useful ways of interpreting, analysing, and solving, linear programming problems and linear complementarity problems arising in physical situations.

The paper is organized as follows. In section 2 we dispose of some preliminaries. In section 3 we introduce the linear program (LP), the dual linear program (LD), and the least

element problem (LE). In section 4 we introduce the linear complementarity problem (LC), the variational inequality (VI), and the unilateral minimization problem (or quadratic programming problem) (UM). In section 5 we discuss the relationship between the linear program, the least element problem, and the variational inequality. In section 6 we discuss in detail a one-dimensional problem. Finally, in section 7, we apply our results to the problem of lubrication cavitation in journal bearings.

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## 2. Preliminaries

X denotes a real Hilbert space with norm $\|\cdot\|$ and dual $Y = X^*$. The evaluation of a continuous linear functional $\ell \in X^*$ at a point $x \in X$ is denoted by $\langle x, \ell \rangle$.

It is assumed that X is partially ordered by a vector ordering $\geq$. Let

$$P = \{x \in X : x \geq 0\} \ .$$

Then (Kelley and Namioka [1976, p. 224]) P is a convex cone in X with vertex at the origin: that is, $P + P \subset P$ and $\lambda P \subset P$ for all non-negative real $\lambda$. We assume that P is closed. $x \geq y$ iff $x - y \geq 0$, that is, $x - y \in P$.

The dual cone $P^* \subset X^*$ is defined by

$$P^* = \{x^* \in X^* : \langle x, x^* \rangle \geq 0 \text{ for all } x \in P\} \ . \tag{2.1}$$

We write $x^* \geq 0$ if $x^* \in P^*$. Since P is closed it follows from the Hahn-Banach theorem that $x \geq 0$ iff $\langle x, x^* \rangle \geq 0$ for all $x^* \in P^*$.

It is also assumed that X is a vector lattice (Kelley and Namioka [1976, p. 229]). That is, for all $x, y \in X$, there exists a unique element $\sup(x,y) \in X$ such that $\sup(x,y) \geq x$ and $\sup(x,y) \geq y$; furthermore, if $z \in X$ satisfies $z \geq x$ and $z \geq y$ then $z \geq \sup(x,y)$. The assumption that X is a vector lattice has the following consequences. For all $x, y \in X$ there exists a unique element $\inf(x,y)$ such that $x \geq \inf(x,y)$ and $y \geq \inf(x,y)$; furthermore, if $z \in X$ satisfies $z \leq x$ and $z \leq y$ then $z \leq \inf(x,y)$. If $x \geq y$ then $\sup(x,y) = x$, and if $y \geq x$ then $\sup(x,y) = y$; since $\sup(x,y)$ is unique, it follows that if $x \geq y$ and $y \geq x$ then $x = y$. For every $x \in X$, $x = \sup(x,0) - \inf(x,0)$ so that $X = P - P$. If $0 = x + y$ where $x, y \in P$ then $x = y = 0$; thus 0 is an extreme point of P, that is, P is a pointed cone.

$M : X \rightarrow Y = X^*$ denotes a continuous linear transformation with adjoint $M^* : Y^* \rightarrow X^*$ defined by

$$\langle x, M^* y^* \rangle = \langle Mx, y^* \rangle \ . \tag{2.2}$$

Associated with M we have the continuous bilinear operator $a : X \times X \rightarrow R^1$ defined by

$$a(v,u) = \langle u, Mv \rangle \ ; \tag{2.3}$$

a   is symmetric if   $a(u,v) = a(v,u)$,   and coercive if

$$a(x,x) \geq \alpha \|x\|^2 \ , \tag{2.4}$$

for some real strictly positive constant   $\alpha$   and all   $x \in X$.

We will sometimes impose the following conditions upon   a   and   M:

Condition S.   If   $r \in X^*$   and   $u,v \in X$   are such that

$\qquad a(u,\psi) \geq \langle \psi,r \rangle$   and   $a(v,\psi) \geq \langle \psi,r \rangle$   for all   $\psi \in P$,

$\qquad$ and if   $w = \inf(u,v)$,   then

$\qquad a(w,\psi) \geq \langle \psi,r \rangle$   for all   $\psi \in P$.   $\square$

Condition Z.   If   $u,v \in P$   satisfy   $\inf(u,v) = 0$,

$\qquad$ then   $a(u,v) \leq 0$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

$\qquad$ p   and   q   denote elements of   $X^*$.   We assume throughout that   $p \in P^*$.   We will some-
time assume that   p   is strictly positive, that is, if   $x \in P$   and   $\langle x,p \rangle = 0$   then   $x = 0$.

$\qquad$ Since

$$a(u,\psi) - \langle \psi,r \rangle = \langle \psi, \ Mu - r \rangle \ , \tag{2.5}$$

Condition S may be rewritten as follows:   if   $Mu \geq r$, $Mv \geq r$,   and   $w = \inf(u,v)$,   then
$Mw \geq r$.   If -M is the Laplacian operator   $\nabla^2$   and   $r = 0$   then   $Mu \geq 0$   means, in an appro-
priate sense, that   -u   is subharmonic.   In this case, Condition S reduces to the well-known
fact that the infimum of two superharmonic functions is superharmonic.   There is, therefore,
a close connection between some of the present results and the theory of subharmonic func-
tions (Rado [1972], Brelot [1945, 1965], Stampacchia [1965], Littman [1963], Moreau [1971]).

$\qquad$ In the case when   M   is a square matrix, Condition Z is equivalent to the requirement
that the off-diagonal elements of   M   be non-positive - that is, that   M   is a   Z-matrix
(Fiedler and Ptak [1962]).   There is, therefore, also a close connection between some of the
present results and the theory of M-matrices and Z-matrices (Plemmons [1976]).   Condition Z
was implicitly used by Stampacchia [1969, p. 151] with the conclusion   $a(u,v) \leq 0$   replaced
by   $a(u,v) = 0$.

$\qquad$ Conditions S   and   Z   are not equivalent because, as shown in section 2.1, the necessary
and sufficient conditions for Conditions S   and   Z   are not equivalent in the case of
matrices.   However, we do have

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## Theorem 2.1

Let $a$ be coercive and satisfy Condition Z. Then $a$ satisfies Condition S.

**Proof:** Let $u, v \in X$ and $r \in X^*$ satisfy $a(u, \psi) \geq \langle \psi, r \rangle$ and $a(v, \psi) \geq \langle \psi, r \rangle$ for all $\psi \geq 0$. We wish to show that if $w = \inf(u, v)$ then $a(w, \psi) \geq \langle \psi, r \rangle$ for all $\psi \geq 0$. To do so, we modify an argument of Stampacchia [1965, p.205].

Introduce the set $U \subset X$ which consists of all $\zeta \in X$ satisfying $\zeta \geq w$. $U = P + w$ is closed and convex. From the fundamental theorem on variational inequalities (Stampacchia [1964]) we know that there exists $\eta \in U$ such that

$$a(\eta, z - \eta) \geq \langle z - \eta, r \rangle, \tag{2.6}$$

for all $z \in U$. In particular, choosing $z = \eta + \psi$, we see that $a(\eta, \psi) \geq \langle \psi, r \rangle$ for all $\psi \geq 0$. The theorem will therefore be proved if we can show that $\eta = w$.

Set $\zeta = \inf(\eta, u) \in U$. From (2.6) with $z = \zeta$,

$$a(\eta, \zeta - \eta) \geq \langle \zeta - \eta, r \rangle . \tag{2.7}$$

On the other hand we know that $\eta - \zeta \geq 0$, $u - \zeta \geq 0$, and $\inf(\eta - \zeta, u - \zeta) = \inf(\eta, u) - \zeta = 0$. Invoking Condition Z we see that

$$
\begin{aligned}
a(\zeta, \zeta - \eta) &= a(u, \zeta - \eta) + a(\zeta - u, \zeta - \eta) , \\
&= a(u, \zeta - \eta) + a(u - \zeta, \eta - \zeta) , \\
&\leq a(u, \zeta - \eta) , \\
&\leq \langle \zeta - \eta, r \rangle .
\end{aligned}
\tag{2.8}
$$

Combining (2.7) and (2.8) we find that

$$a(\zeta - \eta, \zeta - \eta) \leq 0 .$$

Since $a$ is coercive it follows that $\zeta = \inf(u, \eta) = \eta$, so that $\eta \leq u$. Similarly $\eta \leq v$. Hence $\eta \leq \inf(u, v) = w$. But $\eta \in U$ so that $\eta \geq w$. We conclude that $\eta = w$. □

In the case when $M$ is a real square matrix, it is readily shown from Theorem 2.1.1 below that if $a$ is coercive and satisfies Condition S then $a$ satisfies Condition Z. We do not know whether this is true in general.

We now give three examples of spaces and operators fitting into the above framework.

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## 2.1  Example 1

Let $X = Y = X^* = Y^* = R^n$; $M = (m_{ij})$ an $n \times n$ real matrix; and $p = (p_i)$, $q = (q_i)$, $n$-vectors. Let $P$ be the set of vectors in $R^n$ with non-negative components, so that $P$ is closed and $P^* = P$.

$P$ has the additional important property that it has non-empty interior.

Clearly, $p$ is strictly positive iff $p_i > 0$ for all $i$.

It is readily seen that Condition Z is satisfied iff $m_{ij} \leq 0$ for $i \neq j$ (that is, $M$ is a Z-matrix).

**Theorem 2.1.1:** Condition S is satisfied iff every row of $M$ has at most one strictly positive coefficient, that is, $M^T$ is pre-Leontief (Cottle and Veinott [1972, p. 244]).

**Proof:** We first observe that $M^T$ is pre-Leontief iff each row $k$ of the inequality $Mu \geq r$ can be written in the form

$$c_s u_s \geq r_k + \sum_{j=1}^{n} d_j u_j , \qquad (*)$$

where $d_j \geq 0$ for all $j$; $c_s \geq 0$; and where the dependence upon $k$ of $c_s$ and $d_j$ has been suppressed.

First let us assume that $M^T$ is pre-Leontief and that $Mu \geq r$, $Mv \geq r$. Then inequality $(*)$ holds for $u$, and a similar inequality holds for $v$. Since $d_j \geq 0$ we have that if $w = \inf(u,v)$ then

$$c_s w_s = \inf(c_s u_s , c_s v_s) ,$$

$$\geq r_k + \sum_{j=1}^{n} d_j \inf(u_j, v_j) ,$$

$$= r_k + \sum_{j=1}^{n} d_j w_j ,$$

so that Condition S is satisfied.

Now let us assume that Condition S holds but that $M^T$ is not pre-Leontief. Then there is a row $k$ of $M$ with at least two positive coefficients, $m_{ks}$ and $m_{kt}$ say. Thus, the $k$-th row of the inequality $Mu \geq r$ takes the form

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$$m_{ks}u_s \geq r_k - m_{kt}u_t - \sum_{\substack{j=1 \\ j \neq s,t}}^{n} m_{kj}u_j \ ,$$

and a similar inequality holds for $v$. Set $u_s = 1/m_{ks}$, $u_t = -1/m_{kt}$, $v_s = 2u_s$, $v_t = 2u_t$, and $u_j = v_j = 0$ otherwise. Finally, set $w = \inf(u,v)$ and

$$r_j = \min[\sum_{\ell=1}^{n} m_{j\ell}u_\ell \ , \ \sum_{\ell=1}^{n} m_{j\ell}v_\ell] \ ,$$

for all $j$. Then $Mu \geq r$, and $Mv \geq r$. But,

$$\sum_{\ell=1}^{n} m_{k\ell}w_\ell = m_{ks}u_s + m_{kt}v_t + \sum_{\substack{j=1 \\ j \neq s,t}}^{n} m_{kj}u_j \ ,$$
$$= r_k - 1 \ ,$$

so that the inequality $Mw \geq r$ does not hold.

$\square$

In conclusion, we note that if the problems in Examples 2 and 3 below are discretized, the resulting finite difference or finite element matrices usually satisfy Conditions S and Z.

## 2.2 Example 2

Let $X = H_0^1(\Omega)$, where $\Omega$ is a bounded domain (open connected set) in $R^n$, and $H_0^1(\Omega)$ is the Sobolev space of once-differentiable functions vanishing on $\partial\Omega$ (Adams [1975]). Then $Y = X^* = H^{-1}(\Omega)$. Let $M$ be the linear self-adjoint operator,

$$(Mu) \ (t) = -\sum \frac{\partial}{\partial t_j}(a_{ij}(t) \ \frac{\partial u}{\partial t_i}(t)) , \ t \in \Omega \ , \tag{2.2.1}$$

with coefficients $a_{ij}(t)$ which are continuously differentiable, where the indices $i$ and $j$ are summed from $1$ to $n$. It is assumed that $-M$ is uniformly elliptic, so that

$$\sum a_{ij}(t)\xi_i\xi_j \geq \alpha|\xi|^2 \ , \quad t \in \Omega \ , \tag{2.2.2}$$

for all $\xi = (\xi_i) \in R^n$, and some constant $\alpha > 0$.

Every $x \in H_0^1$ has a representation as a measurable function $x(t)$, and any two such representations of $x$ differ only on a set of measure zero. We write $x \geq 0$ if $x(t) \geq 0$ a.e. (almost everywhere). $P = \{x \in X : x \geq 0\}$ is clearly convex.

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To show that $P$ is closed, let $\{x_n\}$ be a sequence of points in $P$ which converges to $x \in H_0^1$. Then $x_n(t)$ converges to $x(t)$ in $L^2(\Omega)$, from which it follows that $x_n(t) \to x(t)$ a.e. Hence, $x(t) \geq 0$ a.e. so that $x \in P$.

$H_0^1$ is a vector lattice: if $x,y \in H_0^1$ then the functions

$$\sup(x,y)(t) = \sup(x(t), y(t)), \quad , \quad t \in \Omega \ ,$$

$$\inf(x,y)(t) = \inf(x(t), y(t)), \quad , \quad t \in \Omega \ , \tag{2.2.3}$$

are representations of elements in $H_0^1$. (Lewy and Stampacchia [1969, p. 169] prove that $H^1(\Omega)$ is a vector lattice, and their proof can be readily adapted to the present case.)

Another very useful property of $H_0^1$ is that if $x \in H_0^1$ and $F$ is a measurable subset of $\Omega$ on which $x(t)$ is constant then (Lewy and Stampacchia [1969, p. 169]),

$$\int_F |\text{grad } x(t)|^2 dt = 0 \ . \tag{2.2.4}$$

As defined in (2.2.1), the operator $M$ can only be applied to functions $u$ which are twice differentiable. Let $a : H_0^1 \times H_0^1 \to R^1$ be the symmetric coercive bilinear operator defined by

$$a(u,v) = \sum \int_\Omega a_{ij}(t) \frac{\partial u}{\partial t_i} \frac{\partial v}{\partial t_j} dt \ . \tag{2.2.5}$$

We extend the domain of definition of $M$ by regarding $M$ as the mapping from $X = H_0^1$ to its dual space $X^* = H^{-1}$ defined by

$$\langle v, Mu \rangle = a(u,v), \quad \text{for all} \quad u,v \in H_0^1 \ . \tag{2.2.6}$$

The standard theory of elliptic operators allows us to assert that $M$ is uniquely defined by (2.2.6) and that $M$ is a homeomorphism of $X = H_0^1$ onto $X^* = H^{-1}$ (Lions and Magenes [1972, p.207]). In particular, $M$ is an open mapping.

### Theorem 2.2.1

$M$ satisfies Conditions S and Z.

**Proof:** To prove Condition Z, let $u,v \in P$ and $\inf(u,v) = 0$. Let $u$ vanish on $F \subset \Omega$ and $v$ vanish on $G \subset \Omega$. Then, using (2.2.4), we conclude that

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$$a(u,v) = \sum \int_\Omega a_{ij}(t)\, \frac{\partial u}{\partial t_i}\, \frac{\partial v}{\partial t_j}\, dt \quad ,$$

$$= \sum \int_F a_{ij}(t)\, \frac{\partial u}{\partial t_i}\, \frac{\partial v}{\partial t_j}\, dt + \int_{G-F} a_{ij}(t)\, \frac{\partial u}{\partial t_i}\, \frac{\partial v}{\partial t_j}\, dt = 0 \quad .$$

so that Condition Z is satisfied.

Stampacchia [1965, p. 205] proves that Condition S is satisfied.

□

We conclude by making some additional remarks about inequalities in $X$ and $X^*$.

(a) Let $x \in H_0^1(\Omega)$. Then $\underline{x \geq 0}$ in the sense of $H^1(\Omega)$ if there exists a sequence $\{\varphi_m\}$ of functions $\varphi_m \in C^1(\overline{\Omega})$ which satisfy $\varphi_m(t) \geq 0$ in $\Omega$ and which converge to $x$ in $H^1(\Omega)$ (Lewy and Stampacchia [1969, p. 155]). If $x \geq 0$ in the sense of $H^1(\Omega)$ then it follows immediately that $x(t) \geq 0$ a.e. . Conversely, let $x \in H_0^1(\Omega)$ satisfy $x(t) \geq 0$ a.e. . If $\hat{x}$ denotes the extension of $x$ to $R^n$ obtained by setting $\hat{x}(t) = 0$ for $t \notin \Omega$, we know that $\hat{x} \in H_0^1(R^n)$ (Adams [1975, p. 57]). The averaged functions $\hat{x}_h$ are smooth and non-negative, and they converge to $\hat{x}$ in $H^1(R^n)$ (Adams [1975, p. 52]). If $\varphi_h = \hat{x}_h|\Omega$ then $\varphi_h \to x$ in $\Omega$, and we can conclude that $x \geq 0$ in the sense of $H^1(\Omega)$. We have thus shown that if $x \in H_0^1(\Omega)$ then $x \geq 0$ in the sense of $H^1(\Omega)$ iff $x(t) \geq 0$ a.e. . This is of importance to us because Stampacchia and his colleagues use $\geq 0$ in the sense of $H^1(\Omega)$.

(b) If $r \in H^{-1}$ is non-negative then there exists a non-negative Borel measure $\mu$ such that

$$\langle \varphi, r \rangle = \int_\Omega \varphi\, d\mu$$

for every $\varphi \in C_0^\infty(\Omega)$ (Schwartz [1973, p. 29]). This is a very elegant characterization of non-negative functionals; unfortunately, it is difficult to apply because its use involves measure theory.

2.3 Example 3

This example is the special case of Example 2 when $\Omega = (a,b) \subset R^1$.

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All the properties of Example 2 remain valid. There is one additional property which is very important: Every $x \in H_0^1(a,b)$ can be represented as a function $x(t)$ which is absolutely continuous on $[a,b]$ and vanishes at the endpoints. (Smirnov [1964]).

If $x \in H_0^1(a,b)$ then $x = x_\ell + x_m + x_r$ where $x_\ell$ vanishes outside a neighborhood of $a$, $x_r$ vanishes outside a neighborhood of $b$, and $x_m$ has compact support in $(a,b)$. The averaged functions $x_m^{(h)}$ of $x_m$ are smooth, have compact support, and converge to $x_m$, as $h \to 0$. We can construct similar approximations $x_\ell^{(h)}$ and $x_r^{(h)}$ to $x_\ell$ and $x_r$ by first translating and then averaging. If $x \geq 0$ a.e. then $\varphi_h = x_\ell^{(h)} + x_m^{(h)} + x_r^{(h)}$ converges to $x$ and we see that $x \geq 0$ in the sense of $H^1(a,b)$.

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3. **The linear program, the dual linear program, and the least element problem.**

With the notation of section 2, the __linear program__ (LP) is:

(LP)     $$\text{Minimize } \langle x, p \rangle \text{ subject to } Mx + q \geq 0 \text{ ,} \qquad\qquad (3.1)$$
$$x \in P$$

The dual program (LDF) which is (formally) dual to LP is:

(LDF)     $$\text{Maximize } (-q, y^*) \text{ subject to } -M^* y^* + p \geq 0 \text{ ,} \qquad\qquad (3.2)$$
$$y^* \in P^{**}$$

where

$$P^{**} = \{ y^* \in Y^* : y^* \geq 0 \} \text{ ,}$$
$$\qquad\qquad\qquad\qquad (3.3)$$
$$= \{ y^* \in Y^* : \langle u^*, y^* \rangle \geq 0 \text{ for all } u^* \in P^* \} \text{ .}$$

If  $x$  is a solution of LP and  $y^*$  is a solution of LDF then,

$$(x, p) - (-q, y^*) = \langle x, -M^* y^* + p \rangle + (Mx + q, y^*) \geq 0 \text{ ,} \qquad\qquad (3.4)$$

so that the value of LP is always greater than or equal to the value of LDF. In particular, if  $\langle x, p \rangle + \langle q, y^* \rangle = 0$  for some feasible  $x$  and  $y^*$  then  $x$  and  $y^*$  are optimal. It may, however, occur that the two values are never equal, in which case there is a duality gap.

Since  $X$  is reflexive we know (Dunford and Schwartz [1966, p. 66]) that there is an isometric isomorphism  $\kappa$  which maps  $X$  onto  $X^{**} = Y^*$  and which is defined by

$$(x, x^*) = (x^*, \kappa x) \text{ .} \qquad\qquad (3.5)$$

Let
$$y^* = \kappa y \text{ ,} \qquad\qquad (3.6)$$

where  $y \in X$  (not Y), so that

$$(-q, y^*) = (-y, q) \text{ .} \qquad\qquad (3.7)$$

We assert that  $y^* \geq 0$  iff  $y \in P$ . First assume that  $y \in P$ . Then, for any  $u^* \in P^*$  $(y, u^*) \geq 0$ , so that  $y^* \geq 0$ . On the other hand, suppose that  $y^* \geq 0$  but that  $y \notin P$ . Then, since the singleton  $\{y\}$  is compact and the cone  $P = \{x \in X : x \geq 0\}$  is closed and convex, these two sets can be separated (Dunford and Schwartz [1966, p. 417]). That is, there exists a linear functional  $f \in Y$  and constants  $\epsilon > 0$  and  $c$  such that

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$$\langle x, f \rangle \geq c \qquad \text{if } x \in P ,$$

$$\langle y, f \rangle \leq c - \epsilon .$$

Using the properties of $P$ we conclude that $c = 0$, so that $f \in P^*$ and $\langle y, f \rangle \leq -\epsilon$. But $\langle y, f \rangle = \langle f, y^* \rangle \geq 0$, and we have a contradiction.

Finally, for any $u \in X$,

$$
\begin{aligned}
\langle u, M^* y^* \rangle &= \langle Mu, y^* \rangle , && \text{(definition of } M^*) \\
&= \langle Mu, \kappa y \rangle , && \text{(equation (3.6))} , \\
&= \langle y, Mu \rangle , && \text{(definition of } \kappa) \\
&= \langle u, \tilde{M} y \rangle ,
\end{aligned}
\qquad (3.8)
$$

where we define the linear operator $\tilde{M} : X \to X^* = Y$ by

$$a(u,v) = \langle v, Mu \rangle = \langle u, \tilde{M} v \rangle . \qquad (3.9)$$

Thus, $M^* y^* + p \in P^*$ iff $\tilde{M} y + p \in P^*$.

Summing up, we see that $y^*$ satisfies LDF iff $y^* = \kappa y$ where $y$ solves:

(LD)        Maximize $\langle -y, q \rangle$ subject to   $-\tilde{M} y + p \geq 0$ ,
            $y \in X$         $y \geq 0$ ,

and we will take this to be the dual of LP in our further work.

Since $X$ is partially ordered, we may also consider the least element problem (LE): Find $x \in P$ such that $Mx + q \geq 0$ and $x \leq u$ for every $u \in P$ satisfying $Mu + q \geq 0$. LE has at most one solution, for if $x_1$ and $x_2$ were two solutions we would have $x_1 \leq x_2$ and $x_2 \leq x_1$ which implies that $x_1 = x_2$.

In the special case $X = R^n$, there exists a very satisfactory theory for LP and LD, and Mangasarian [1976] used this as the starting point for his study of the relationship between LP and LCP (the linear complementarity problem). LE has also been studied in the finite dimensional case (Cottle and Veinott [1972]).

The case when $X$ is infinite dimensional is much more difficult. It is usually assumed, for example by Ekeland and Temam [1974, p. 66], that the Arrow-Hurwicz constraint qualification is satisfied, namely that there exists $u \in P$ such that $Mu + q$ is an interior point

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of $P^*$. An example of Craven [1977, p. 331] illustrates the difficulties which can arise when $P^*$ does not have any interior points and when $M$ is not an open map. Dempster [1975] develops a general framework for the analysis of LP and LD.

In the present paper we prove the existence of solutions to LP and LE by using the theory of variational inequalities. We do not prove the existence of a solution to LD, although in section 6 we give an example in which LD does have a solution.

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4. **The linear complementarity problem, the variational inequality, and the unilateral**

**minimization problem.**

The <u>linear complementarity problem</u> (LC) is as follows: Find $x \in P$ such that

(LC) $\qquad\qquad\qquad Mx + q \geq 0, \quad (x, Mx + q) = 0$ . $\qquad\qquad\qquad$ (4.1)

The <u>variational inequality</u> (VI) is: Find $x \in P$ such that

(VI) $\qquad\qquad\qquad a(x, v - x) + (v - x, q) \geq 0$ , $\qquad\qquad\qquad$ (4.2)

for all $v \in P$.

If $a$ is symmetric then the <u>unilateral minimization problem</u> (or quadratic programming problem) (UM) is: Find $x \in P$ such that

(UM) $\qquad\qquad\qquad J(x) \leq J(u), \quad$ for all $u \in P$ $\qquad\qquad\qquad$ (4.3)

where

$$J(u) = a(u,u) + 2(q,u) .$$ $\qquad\qquad\qquad$ (4.4)

The basic result on variational inequalities is due to Stampacchia [1964]: if $a$ is coercive then there exists a unique solution to VI.

The connection between VI and UM was also observed by Stampacchia [1964]: if $a$ is symmetric and coercive, then VI is equivalent to UM.

The relationship between VI and LC was noted independently by a number of workers including Lions and Stampacchia [1969, p. 172], Karamardian [1971], Moré [1971]. The basic result, which we prove for the convenience of the reader is:

**Theorem 4.1**

LC is equivalent to VI.

Proof: Assume first that $x$ solves LC. Then for any $v \in P$

$$a(x, v - x) + (v - x, q) = (v - x, Mx + q) ,$$
$$= (v, Mx + q) \geq 0 ,$$

so that $x$ solves VI.

Now assume that $x$ solves VI. Then setting first $v = 0$ and then $v = 2x$ we see that $(-x, Mx + q) \geq 0$ and $(x, Mx + q) \geq 0$, from which we conclude that $(x, Mx + q) = 0$.

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But then, $a(x, v - x) + (v - x, q) = (v, Mx + q) \geq 0$ for all $v \geq 0$, so that $Mx + q \geq 0$, and hence $x$ solves LC.

□

5. The relationship between the linear program, the least element problem, and the linear complementarity problem.

Theorem 5.1

If  a  is coercive and satisfies Condition Z, then LE has a solution, namely the unique solution of VI.

Proof: The proof is a modification of proof of Stampacchia [1969, p. 151] who implicitly used Condition Z in the special form: if  $u,v \in P$  and  $\inf(u,v) = 0$  then  $a(u,v) = 0$.

Let  u  be the unique solution of VI so that  $u \in P$  and

$$a(u, v - u) + \langle v - u, \alpha \rangle \geq 0$$

for all  $v \in P$.

In particular, choosing  $v = u + w$  for any  $w \in P$  we conclude that  $Mu + q \geq 0$.

Now let  w  be any element such that  $w \in P$  and  $Mw + q \geq 0$. We assert that  $w \geq u$. To see this, let  $\zeta = \min(u,w) \in X$,  so that  $w - \zeta \geq 0$  and  $u - \zeta \geq 0$. Furthermore,  $\inf(w - \zeta, u - \zeta) = \inf(w,u) - \zeta = 0$.

Then

$$a(u - \zeta, u - \zeta) = [a(\zeta, \zeta - u) + \langle \zeta - u, q \rangle] - [a(u, \zeta - u) + \langle \zeta - u, q \rangle] \quad ,$$

$$\leq [a(\zeta, \zeta - u) + \langle \zeta - u, q \rangle] \quad ,$$

because  u  satisfies VI. But

$$a(\zeta, \zeta - u) + \langle \zeta - u, q \rangle = a(w - \zeta, u - \zeta) + [a(w, \zeta - u) + \langle \zeta - u, q \rangle] \quad ,$$

$$\leq 0 \quad ,$$

because the first term on the right is nonpositive by Condition Z and the second term is nonpositive since  $Mw + q \geq 0$  and  $\zeta - u \leq 0$.

Combining the above inequalities we see that  $a(u - \zeta, u - \zeta) \leq 0$. Remembering that  a  is coercive we conclude that  $u = \zeta$. Thus,  $w \geq \zeta = u$  so that  u  is a solution of LE.

□

Theorem 5.2

(i) If  x  solves LE then  x  solves LP.

(ii) If  a  satisfies Condition S and  p  is strictly positive, then LP has at most one solution.

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(iii) If $a$ satisfies Condition S, $p$ is strictly positive, and $x$ solves LP then $x$ solves LE.

**Proof:** (i) is obvious. To prove (ii), let $x_1$ and $x_2$ be two solutions of LP. By Condition S, $\zeta = \inf(x_1, x_2) \in P$ satisfies $M\zeta + q \geq 0$ and $(\zeta, p) \leq (x_1, p)$. Since $x$ is optimal, $(\zeta, p) = (x_1, p)$ and we conclude that $\zeta = x_1$. Similarly, $\zeta = x_2$, so that $x_1 = x_2$.

To prove (iii), let $u \in P$ satisfy $Mu - q \geq 0$. Set $\zeta = \inf(u, x)$. Then $M\zeta + q \geq 0$ and $(\zeta, p) = (x, p)$ so that $\zeta = x$. Hence, $u \geq x$ and $x$ solves LE.

□

Remembering that if $a$ is coercive and $a$ satisfies Condition Z then $a$ satisfies Condition S (Theorem 2.1) we find that

## Theorem 5.3

If $a$ is coercive and satisfies Condition Z, and if $p$ is strictly positive, then LP, LE, VI, and LC all have the same unique solution.

□

## Theorem 5.4

Assume that $x$ solves VI, that $y$ solves LD, that $(x, p) + (y, q) = 0$, that $a$ is symmetric and coercive and satisfies Condition Z, and that $p + q \geq 0$.

Then $y \geq x$.

**Proof:** Set $w = \inf(x, y)$. Then

$$a(x - w, x - w) = a(x - y, x - w) + a(y - w, x - w),$$
$$\leq a(x - y, x - w) ,$$

since $y - w \geq 0$, $x - w \geq 0$, and $\inf(y - w, x - w) = 0$. But,

$$a(x - y, x - w) = a(x, x - w) - a(y, x - w) ,$$
$$= a(x, x - w) - a(x - w, y) ,$$
$$= a(x, x - w) + (x - w, - \tilde{M}y) ,$$
$$= [a(x, x - w) + (x - w, q)] - (x - w, p + q) +$$
$$+ (x - w, - \tilde{M}y + p) .$$

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The first term on the right is negative because $x$ solves VI. The second term is negative because $p + q \in P^*$ and $x - w \in P$. The third term is zero because the equality

$$0 = \langle x,p \rangle + \langle y,q \rangle = \langle x, - \tilde{M}y + p \rangle + \langle y, Mx + q \rangle$$

implies that $\langle x, - \tilde{M}y + p \rangle = 0$ and hence, since $0 \leq w \leq x$, that $\langle w, - \tilde{M}y + p \rangle = 0$.

Combining the above, we conclude that $a(x - w, x - w) \leq 0$ so that $x = w$. Then $y \geq w = \inf(x,y) = x$.

$\square$

It may be observed that if $x$ solves LP, $y$ solves LD, $\langle x,p \rangle + \langle y,q \rangle = 0$, and $y \geq x$, then we have that

$$0 \leq \langle x, Mx + q \rangle \leq \langle y, Mx + q \rangle = 0 \quad ;$$

that is, $x$ solves LC.

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## 6. A one-dimensional problem

We consider a special case of Example 3 (section 2.3): $X = H_0^1(0,2)$,

$$\text{Min}(x,p) = \int_0^2 1 \cdot x(t)dt \quad \text{subject to} \quad x(t) \geq 0 \quad \text{a.e.,}$$

$$\text{and} \quad Mx + q \equiv -\ddot{x}(t) + (t - 1) \geq 0 \; , \tag{6.1}$$

with the corresponding dual problem

$$\text{Max}(y,-q) = -\int_0^2 (t - 1)y(t)dt \quad \text{subject to} \quad y(t) \geq 0 \quad \text{a.e.,}$$

$$-\tilde{M}y + p \equiv \ddot{y}(t) + 1 \geq 0 \; . \tag{6.2}$$

The inequality $-\ddot{x} + (t - 1) \geq 0$ is interpreted in the sense that

$$\langle \varphi, Mx + q \rangle = \int_0^2 [\dot{x}(t)\dot{\varphi}(t) + (t - 1)\varphi(t)]dt \geq 0 \; , \tag{6.3}$$

for all non-negative $\varphi \in H_0^1(0,2)$, and the inequality $\ddot{y} + 1 \geq 0$ is interpreted in the same way.

This problem was chosen because it is a simple problem with the same general structure as the problem for a cavitating journal bearing which is discussed in the next section.

There is a straightforward procedure for obtaining possible solutions of such one-dimensional problems; these solutions can then be verified a-postiori. We assume that $x(t) > 0$ for $0 < t < \tau$ and $x(t) = 0$ for $\tau \leq t \leq 2$, where $\tau$ is an unknown constant corresponding to the free boundary (the point $t = \tau$). If $x$ also satisfies LC then $(x, -\ddot{x} + (t - 1)) = 0$, so that $-\ddot{x}(t) + (t - 1) = 0$ for $0 \leq t \leq \tau$. The general solution of the equation $-\ddot{x} + (t - 1) = 0$ is

$$x(t) = A + Bt + \frac{1}{6}(t - 1)^3 \; . \tag{6.4}$$

Using the conditions $x(0) = x(\tau) = 0$ to determine the constants $A$ and $B$ we find

$$x(t) = t(t - \tau)[-3 + t + \tau]/6 \; . \tag{6.5}$$

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To determine $\tau$ we note that the condition $-\ddot{x} + (t - 1) \geq 0$ implies that for all smooth non-negative $\varphi \in H^1_0(0,2)$,

$$\langle \varphi, \, Mx + q \rangle = \int_0^2 [\dot{x} \, \dot{\varphi} + (t - 1)\varphi] dt \ ,$$

$$= \int_0^\tau [\dot{x}\dot{\varphi} + (t - 1)\varphi] dt + \int_\tau^2 (t - 1)\varphi \, dt \ ,$$

$$= \dot{x}\varphi \Big]_0^\tau + \int_0^\tau [-\ddot{x}\varphi + (t - 1)\varphi] \, dt + \int_\tau^2 (t - 1)\varphi \, dt \ , \qquad (6.6)$$

$$= \dot{x}(\tau-)\varphi(\tau) + \int_\tau^2 (t - 1)\varphi \, dt \ ,$$

$$\geq 0 \ .$$

This is only possible if $\tau \geq 1$ (so that $t - 1 \geq 0$ for $t \in [\tau, 2]$) and $\dot{x}(\tau-) \geq 0$. But, $x(t) \geq 0$ for $t \leq \tau$ and $x(\tau) = 0$ so $\dot{x}(\tau-) \leq 0$. We conclude that $\dot{x}(\tau-) = \dot{x}(\tau+) = \dot{x}(\tau)$ $= 0$. The condition $\dot{x}(\tau) = 0$ leads to an algebraic equation for $\tau$, namely,

$$\dot{x}(\tau) = \tau[-3 + 2\tau]/6 = 0 \ ;$$

thus, $\tau = 3/2$ and

$$x(t) = t(t - 3/2)^2/6 \ , \quad 0 \leq t \leq 3/2 \ ,$$
$$= 0 \ , \quad 3/2 \leq t \leq 2 \ , \qquad (6.7)$$

is our trial solution.

Using (6.6) and (6.7) we see that $x$ is such that $x \geq 0$, $-\ddot{x} + (t - 1) \geq 0$, and $\langle x, \, Mx + q \rangle = 0$, so that $x$ is a solution of LC. Invoking Theorems 4.1 and 5.3, we conclude that $x$ is the unique solution of LP.

We now consider the determination of $y$. Since $\langle x, \, y + 1 \rangle = 0$, it follows that $\ddot{y}(t) + 1 = 0$ when $x(t) > 0$, that is, when $0 < t < \tau$. On the other hand, since $\langle y, \, -\ddot{x} + (t - 1) \rangle = 0$, it follows that $y(t) = 0$ when $-\ddot{x} + (t - 1) > 0$, that is, when $\tau < t < 2$. We conclude that $\ddot{y}(t) + 1 = 0$ for $0 \leq t \leq 3/2$ and $y(t) = 0$ for $3/2 \leq t \leq 2$. Solving this boundary value problem we obtain

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$$y(t) = t[-2t + 3]/4, \quad 0 \leq t \leq 3/2 \;,$$
$$= 0 \quad, \quad 3/2 \leq t \leq 2 \;. \tag{6.8}$$

The condition $y \geq 0$ is seen to be satisfied.

Direct computation yields

$$(x,p) = \int_0^2 x(t)dt = \frac{9}{128} = -\int_0^2 (t - 1)y(t)dt = -(y,q) \;. \tag{6.9}$$

The solutions $x(t)$ and $y(t)$ are plotted in Figure 6.1. We note that $y \geq x$ as proved in Theorem 5.4.
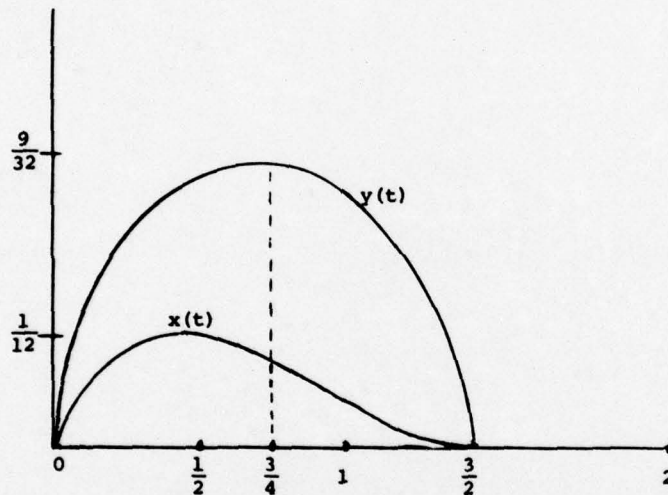


Figure 6.1: $x(t)$ and $y(t)$

It is possible to give two justifications for the free boundary condition $\dot{x}(\tau) = 0$. Firstly, if $x \in H_0^2(0,1)$, as is often the case, then $\dot{x}(t)$ is continuous so that $\dot{x}(\tau) = \dot{x}(\tau+) = 0$. Secondly, a reasonable interpretation of the condition $-\ddot{x}(\tau) + (\tau - 1) \geq 0$ is that

$$\lim_{\Delta t \to 0} - \frac{\dot{x}(\tau + \Delta t) - \dot{x}(\tau - \Delta t)}{\Delta t} + (\tau - 1) \geq 0 \quad .$$

Since $\dot{x}(\tau + \Delta t) = 0$ and $\dot{x}(\tau - 0) \leq 0$, it follows that $\dot{x}(\tau - 0) = 0$.

## 7. Lubrication cavitation of journal bearings

A large number of physical problems can be formulated as linear complementarity problems in which a differential equation (ordinary or partial) must be solved subject to the inequality constraint that the solution be non-negative; roughly speaking, at any point the solution must either be zero or satisfy the differential equation (Cryer [1977, in preparation], Duvaut and Lions [1972]). The reformulation of such linear complementarity problems as linear programs has two advantages: (i) it suggests alternative methods of solving the problems; and (ii) it sometimes provides a physically meaningful interpretation. As an example of such linear complementarity problems we consider here the problem of a cavitating journal bearing.

A journal bearing consists of a circular cylinder (the journal) which is rotating inside a support structure (the bearing). The narrow gap between the journal and the bearing is filled with a thin film of lubricating fluid. Various geometries are possible. In Figure 7.1 we show a partial journal bearing of finite length. The term 'partial' refers to the fact that the journal is not completely enclosed within the bearing, and is partially exposed to the atmosphere.
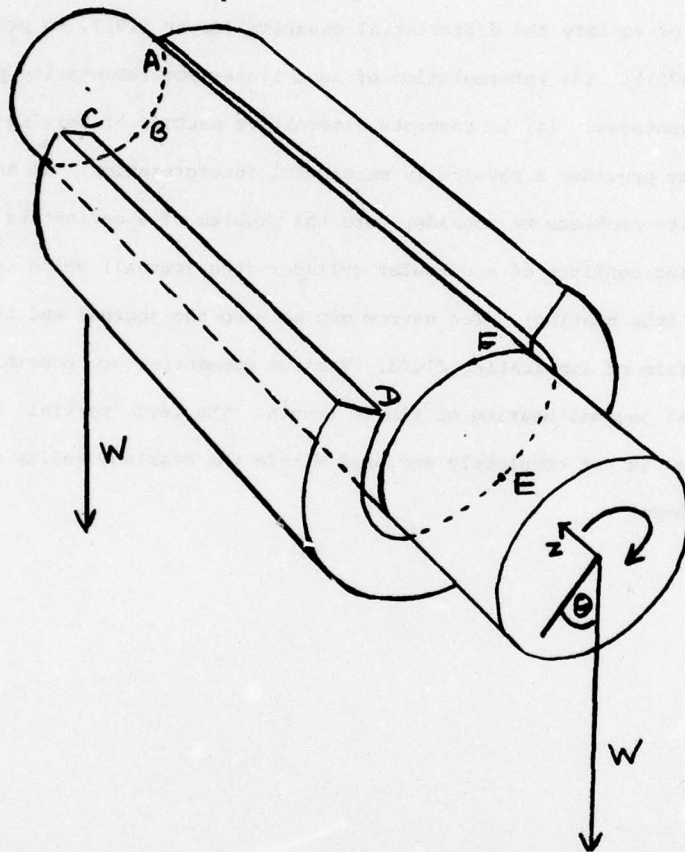
**Figure 7.1:   A partial journal Bearing**

It is required to determine the pressure  x  of the lubricant, and the load  W  borne

by the bearing.  Because the gap between the journal and the bearing is very narrow, the

simplifications of lubrication theory can be applied.  In particular, it is assumed that the

pressure does not vary across the gap, so that the problem becomes a two-dimensional problem

in the rectangular domain  $\Omega$ = ABCDEF  in the  $\theta z$ -plane (Figure 7.2).



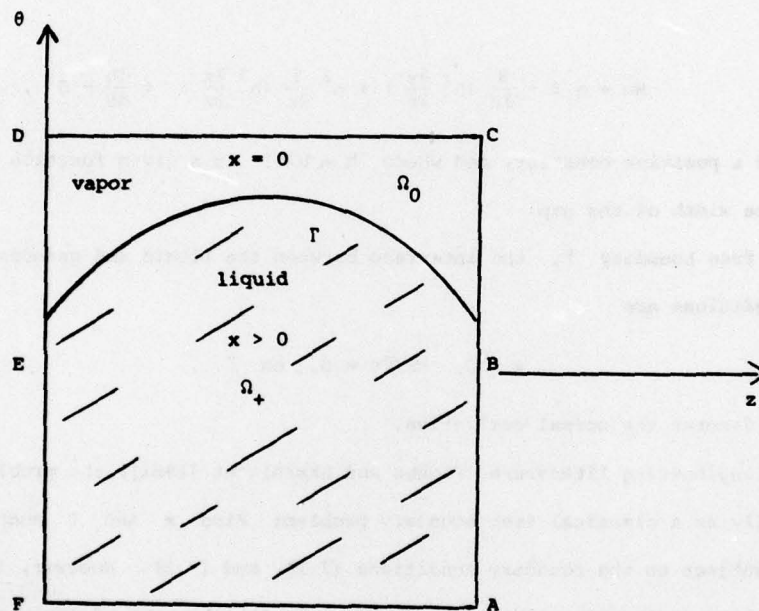Figure 7.2:  The domain  $\Omega$

The lubricant flows in from a reservoir along the entry edge  AF  and flows out

through the ends  ABC  and  DEF  as well as through the exit edge  CD.  At all these points

the lubricant is in contact with the atmosphere, and if the pressure is normalized so that

atmospheric pressure is zero, then the boundary conditions are that  x = 0  on  $\partial\Omega$.  That is,

$$x \in X = H_0^1(\Omega) \quad . \tag{7.1}$$

The lubricant occurs in both liquid and gaseous phases. It is assumed that the lubricant vaporizes when the pressure is zero, so that the inequality $x \geq 0$ must be satisfied everywhere. If the pressure is greater than zero then the lubricant is in the liquid phase and satisfies the simplified form of the Navier-Stokes equations known as Reynolds' equation. After introducing dimensionless variables, the equation takes the form (Pinkus and Sternlicht [1961]):

$$Mx + q \equiv - \frac{\partial}{\partial \theta} \left( h^3 \frac{\partial x}{\partial \theta} \right) + \alpha^2 \frac{\partial}{\partial z} \left( h^3 \frac{\partial x}{\partial z} \right) + \frac{dh}{d\theta} = 0 \quad , \tag{7.2}$$

where $\alpha$ is a positive constant, and where $h = h(\theta)$ is a given function which is proportional to the width of the gap.

On the free boundary $\Gamma$, the interface between the liquid and gaseous phases, the boundary conditions are

$$x = 0, \quad \partial x / \partial n = 0, \quad \text{on } \Gamma \quad , \tag{7.3}$$

where $\partial / \partial n$ denotes the normal derivative.

In the engineering literature (Pinkus and Sternlicht [1961]) the problem is formulated mathematically as a classical free boundary problem: Find $x$ and $\Gamma$ such that $x$ satisfies (7.2) subject to the boundary conditions (7.1), and (7.3). However, in a large number of papers in the engineering literature, beginning with the work of Christopherson [1941], numerical approximations have been obtained in a completely different way: equation (7.2) is replaced by finite differences, and the resulting system of algebraic equations is solved as a finite-dimensional linear complementarity problem (Cryer [1971]) which may be considered as a discretization of the infinite-dimensional linear complementarity problem

$$x \geq 0, \quad Mx + q \geq 0, \quad (x, Mx + q) = 0 \quad . \tag{7.4}$$

We may thus take (7.4) as the starting point for a mathematical analysis of the problem. The problem is a special case of Example 2 (section 2.2), and it follows from Theorem 5.3

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that there exists a unique solution $x \in H_0^1(\Omega)$ of LP, LE, VI, and LC.

In the engineering literature, there has been some discussion of an appropriate variational principle for the problem (Christopherson [1957]). The formulation as a variational inequality leads to two useful variational principles:

(1) Since $a$ is symmetric, the problem is equivalent to the unilateral minimization problem

$$\text{Inf}_{v \geq 0} J(v) = a(v,v) + 2\langle v,q \rangle \quad .$$

(2) For any strictly positive function $p(\theta,z)$, the problem is equivalent to the linear programming problem

$$\text{Min} \langle x,p \rangle \equiv \int_\Omega x(\theta,z) p(\theta,z) d\theta \, dz \quad ,$$

subject to $x \geq 0$, $Mx + q \geq 0$. In particular, if $-\pi/2 < \theta_F < \theta_D < \pi/2$ (see Figures 7.1 and 7.2), then $p = \cos \theta > 0$ and $\langle x,p \rangle$ is the load $W$ borne by the bearing in the vertical direction (Figure 7.1). That is, the solution $x$ minimizes the vertical load.

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Linear complementarity problem; linear program; variational inequality; lubrication cavitation; partially ordered Hilbert space.

**20. ABSTRACT (Continue on reverse side if necessary and identify by block number)**

Let $X$ be a vector lattice Hilbert space with dual $X^*$. Let $M$ be a continuous linear mapping of $X$ onto $X^*$. Let $p, q \in X^*$ with $p > 0$. We consider the relationship between the <u>linear complementarity problem</u>: Find $x \in X$ such that $x \geq 0$, $Mx + q > 0$, $\langle x, Mx + q \rangle = 0$, and the <u>linear programming problem</u>: Find $x \in X$ which minimizes $\langle x, p \rangle$ subject to $x \geq 0$, $Mx + q \geq 0$.

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